*Int. J. Multiphase Flow*, Vol. 2, pp. 79–95. Pergamon/Elsevier, 1975. Printed in Great Britain.

# LIQUID VELOCITY DISTRIBUTION IN TWO-PHASE BUBBLE FLOW

Y. SATO

Mechanical Engineering Department, Kumamoto University, Kumamoto, Japan

and

K. SEKOGUCHI

Chemical Engineering Department, Kyushu University, Fukuoka, Japan

(*Received* 31 *October* 1973)

**Abstract**—An improved analytical treatment is developed which makes possible the satisfactory prediction of the liquid velocity distribution in two-phase bubble flow.

In the analysis, the shear stress in the liquid phase is regarded as important. When the fluctuation of turbulent velocity can be subdivided into two components: one due to the inherent liquid turbulence independent of the existence of the bubble, $(u', v')$, and the other due to the additional liquid turbulence by the bubble agitation, $(u'', v'')$, it is possible to split the shear stress into two components, $-\rho\overline{u'v'}$ and $-\rho\overline{u''v''}$ corresponding to $(u', v')$ and $(u'', v'')$, respectively.

A basic equation for the liquid velocity distribution is derived from further development of this treatment. The agreement between the measured velocity profiles and those calculated is quite close especially in the core region of a duct.

## 1. INTRODUCTION

The prediction of the liquid velocity distribution in bubble flow has been a subject of work in the area of two-phase gas–liquid flow for the past 10 years. It is believed that knowledge of liquid velocity is concerned with not only an exchange of momentum, but also the transfer of heat and mass in many systems of engineering interest.

Several models have been proposed to describe the velocity distribution. If two-phase mixtures are regarded as a continuous medium, the turbulent exchange of density should be considered as well as that of momentum. On this presumption, Levy (1963) derived a velocity distribution with the aid of the mixing length theory for single-phase flow. Bankoff (1960) analyzed it under the assumption that the shear stress is uniform in a cross-section, and the mixing length is equal to that in the single-phase flow. Koide & Kubota (1966) proposed a treatment similar to Bankoff's except that the mixing length was expressed as a function of the local void fraction. Brown & Kranich (1968) employed the well-known logarithmic velocity distribution law even in bubble flow, neglecting relative velocity between both phases and introducing the arbitrarily defined density and viscosity of the mixture. Beattie (1972) suggested an analysis on the assumption that bubbles are treated as only voidages, the shear stress is uniform in a cross-section, the mixing length is the same as for single-phase flow, and the local void fraction is proportional to the liquid velocity.

The bubble flow regime has a complicated flow mechanism, since the void fraction distribution across a flow channel is influenced by the changes in the size and number of bubbles generated, even at the same flow rates of gas and liquid. The changes in void fraction distribution perhaps yield appreciable effects on liquid velocities.

In a two-phase vertical flow, radial distributions of shear stress strongly depend upon those of average mixture density, which is directly connected to void fraction. It is obvious on physical grounds that shear stress is closely tied to the liquid velocity gradient. Thus, for the satisfactory prediction of liquid velocity, it is necessary to take account of the role of shear stress or void fraction in the analysis. For this reason, each model noted in the foregoing leaves much room for improvement.

The purpose of this paper is to represent a theoretical relationship between liquid velocity and void fraction. In the analysis, a new proposal is offered for introducing an additional shear stress due to bubble motion. Also, comparisons are made between the predictions and measurements on vertical upward air–water bubble flow.

## 2. THEORY

### 2.1 *Flow in a channel with two parallel flat walls*

2.1.1 *Flow system and assumptions.* Consider a steady two-dimensional bubble flow between vertical parallel walls as shown in figure 1. The $x$-axis is parallel to the wall and the $y$-axis normal to it. Liquid velocity components $u$ and $v$ have directions parallel and normal to the wall. It is assumed throughout the present analysis that fluids are incompressible, and the gas phase behaves only as a voidage owing to its discreteness and its negligibly small density. The latter assumption allows us to consider the shear stress in the liquid phase only.

This theory gives a functional relationship between liquid velocity and void fraction. In order to seek a complete solution, i.e. to evaluate analytically the two parameters, one more relation should be introduced. In other words, even though this analysis will be

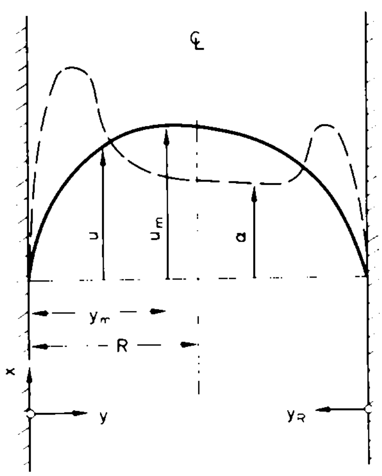


Figure 1. Parameters in two-phase flow.

experimentally validated, another relation is necessary which might pertain to phase distribution. In this paper, the theoretical prediction of liquid velocity is carried out using measured void fraction profiles.

2.1.2 *Additional turbulent stress due to bubble agitation.* The equation of motion in the $x$-direction for two-dimensional single-phase incompressible fluid flow can be rewritten in the form

$$\frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u - g. \qquad [1]$$

Equation [1] is applied to the liquid phase in bubble flow. To describe a turbulent flow in mathematical terms it is usual to separate it into a mean motion and into a fluctuation. The same procedure is also convenient for analysing two-phase bubble flow. Furthermore, it is assumed that the velocity and pressure fluctuations in the liquid can be subdivided further into two components, which are caused by the motion of bubbles relative to the surrounding liquid (i.e. bubble agitation) and by the inherent liquid turbulence independent of the existence of bubbles. Denoting the time-average of the $u$-component of velocity by $u$, and the velocities of fluctuations by $u'$ and $u''$ (independent of and dependent on bubble agitation, respectively), the following relations for the velocity components and pressure can be written

$$u = \bar{u} + u' + u'', \quad v = \bar{v} + v' + v'', \quad p = \bar{p} + p' + p''. \qquad [2]$$

Substituting these variables into [1] and forming its time-average yields

$$\frac{\partial(\bar{u}^2)}{\partial x} + \frac{\partial(\bar{u}\bar{v})}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \frac{\partial}{\partial x}\{\overline{(u' + u'')^2}\} - \frac{\partial}{\partial y}\{\overline{(u' + u'')(v' + v'')}\} - g. \qquad [3]$$

Since the respective fluctuations ($u'$, $v'$) and ($u''$, $v''$) are independent of each other, the correlations of $\overline{u'u''}$, $\overline{u'v''}$ and $\overline{u''v'}$ in the third and fourth terms on the right-hand side of [3] become zero. Therefore, it is seen from the equation that the following additional stresses arise in the turbulent liquid

$$\begin{bmatrix} \sigma'_x + \sigma''_x & \tau'_{yx} + \tau''_{yx} \\ \tau'_{xy} + \tau''_{xy} & \sigma'_y + \sigma''_y \end{bmatrix} = -\begin{bmatrix} \rho(\overline{u'^2} + \overline{u''^2}) & \rho(\overline{u'v'} + \overline{u''v''}) \\ \rho(\overline{u'v'} + \overline{u''v''}) & \rho(\overline{v'^2} + \overline{v''^2}) \end{bmatrix}. \qquad [4]$$

Equation [4] suggests that the shear stress of the two-dimensional bubble flow can be divided into three components as follows:

$$\begin{aligned} \tau &= (1 - \alpha)\left(\mu\frac{\mathrm{d}\bar{u}}{\mathrm{d}y} - \rho\overline{u'v'} - \rho\overline{u''v''}\right) \\ &= \tau_\ell + \tau_t + \tau_B \\ &= \rho(1 - \alpha)(\nu + \varepsilon' + \varepsilon'')\frac{\mathrm{d}\bar{u}}{\mathrm{d}y} \end{aligned} \qquad [5]$$

where $\alpha$ is the local void fraction and where $\tau_\ell$ is the shear stress due to liquid viscosity, $\tau_\ell = (1 - \alpha)\rho\nu(\mathrm{d}\bar{u}/\mathrm{d}y)$. $\tau_t$ is a second component due to the momentum exchange in turbulent flow excluding the effect of bubble agitation, while $\tau_B$ is due to that of bubble agitation. Denoting the eddy diffusivities for both the shear stresses $\tau_t$ and $\tau_B$ by $\varepsilon'$ and $\varepsilon''$, they are written as

$$\tau_t = (1 - \alpha)\rho\varepsilon'\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}, \quad \tau_B = (1 - \alpha)\rho\varepsilon''\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}.$$

In the above equations, shear stress in the liquid phase is taken into account, but shear stress inside bubbles is ignored. The factor $(1 - \alpha)$ means the probability of the existence of liquid phase at a point, which is also the time-averaged liquid volume fraction at the point.

Consider a plane, located at $y = y_1$ from the wall, which is designated as the "control plane" (figure 2). When a bubble passes in the vicinity of a control plane, the bubble gives rise to the so-called drift (Milne-Thomson 1968) on the surrounding liquid. As a result, liquid is transferred to the control plane from both sides, where different mean velocities prevail, and thus additional velocity fluctuations, $(u'', v'')$, are caused. It is expected that fluctuations of this type are closely related to the relative velocity of bubbles to the surrounding liquid $U_B$, the bubble diameter $d_B$ and the distance $\eta$ from the control plane to the center of the bubble.

Since in a real bubbly flow a large number of bubbles of various sizes flow in a channel and are spread over the cross-sectional area, the velocity fluctuations on the control plane result from the effects of the superposition of individual bubbles. A method estimating the resultant velocity fluctuations is proposed in the Appendix, where, analogous to the

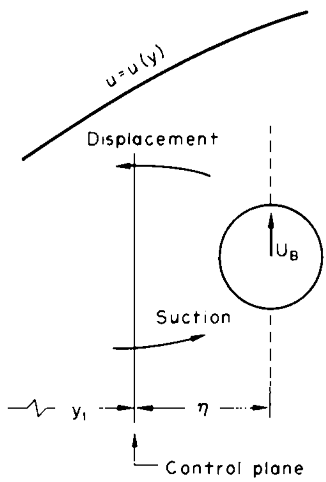


Figure 2. Transport of liquid particle due to bubble motion.

successful treatment in single-phase turbulent flow, the eddy diffusivity due tp bubble $\varepsilon''$ is introduced and expressed as:

$$\varepsilon'' = \kappa_1 \alpha (\hat{d}_B/2)\hat{U}_B \qquad [6]$$

in which $\kappa_1$ is an empirical constant, $\alpha$ the local void fraction, and $\hat{d}_B$ and $\hat{U}_B$ the spatial mean values of the diameter and relative velocity of bubbles respectively. In accordance with this equation the additional turbulent shear stress can be represented as

$$\tau_B = \kappa_1 \rho(1-\alpha)\alpha \frac{\hat{d}_B}{2}\hat{U}_B \frac{du}{dy}. \qquad [7]^*$$

The bubbles in the bubble sublayer† are appreciably different, not only in size but also in relative velocity between bubble and liquid, from those in the core region (Sekoguchi *et al.* 1972b; Sato *et al.* 1973). Therefore, for the region very close to the wall a more complicated procedure may be necessary to predict the shear stress due to bubble motion.

2.1.3 *Total turbulent shear stress in bubble flow.* For the eddy diffusivity in single-phase turbulent flow, Reichardt (1951) proposed the following expression:

$$\frac{\varepsilon'}{\nu} = \frac{\kappa R}{6}\frac{\sqrt{(\tau_w/\rho)}}{\nu}\left\{1 - \left(\frac{r}{R}\right)^2\right\}\left\{1 + 2\left(\frac{r}{R}\right)^2\right\} \qquad [8]$$

where $\kappa$ is the mixing length constant, $R$ the radius of a pipe or the half width of a rectangular channel, and $\tau_w$ the wall shear stress. The above equation is based upon an experiment in a rectangular channel, and agrees with the data especially in the core region, i.e. $r/R < 0.9$. Introducing [8] to the present flow problem, the apparent eddy diffusivity, independent of the existence of bubbles, is written as:

$$\varepsilon' = \frac{\kappa y_m}{6}\sqrt{\left(\frac{\tau_w}{\rho}\right)}\left\{1 - \left(\frac{y_m - y}{y_m}\right)^2\right\}\left\{1 + 2\left(\frac{y_m - y}{y_m}\right)^2\right\} \qquad [9]$$

where $y_m$ denotes the distance measured from the wall to the zero shear plane, on which the velocity gradient becomes zero, i.e. $du/dy = 0$. The corresponding turbulent shear stress can be expressed as

$$\begin{aligned}\tau_t &= \rho(1-\alpha)\varepsilon'\frac{du}{dy} \\ &= \rho(1-\alpha)\frac{\kappa y_m}{6}\sqrt{\left(\frac{\tau_w}{\rho}\right)}\left\{1 - \left(\frac{y_m - y}{y_m}\right)^2\right\}\left\{1 + 2\left(\frac{y_m - y}{y_m}\right)^2\right\}\frac{du}{dy}. \qquad [10]\end{aligned}$$

* From this point onwards the bar above the velocity to denote time-average will be omitted.

† A layer including bubbles which slide on a wall is termed a bubble sublayer. Thus, its thickness is nearly equal to the diameter of the bubbles (Sekoguchi *et al.* 1972a).

If viscous stress $\tau_\ell$ can be neglected when compared with $\tau_t$ and $\tau_B$, [7] and [10] give the total shear stress in two-dimensional bubble flow as follows:

$$\tau = \rho(1 - \alpha)(\varepsilon' + \varepsilon'')\frac{\mathrm{d}u}{\mathrm{d}y}$$

$$= \rho\varepsilon_{TP}\frac{\mathrm{d}u}{\mathrm{d}y} \qquad [11]$$

where

$$\varepsilon_{TP} = (1 - \alpha)\left[\frac{\kappa y_m}{6}\sqrt{\left(\frac{\tau_w}{\rho}\right)}\left\{1 - \left(1 - \frac{y}{y_m}\right)^2\right\}\left\{1 + 2\left(1 - \frac{y}{y_m}\right)^2\right\} + \kappa_1\alpha\left(\frac{\hat{d}_B}{2}\right)\hat{U}_B\right]. \qquad [12]$$

2.1.4 *Derivation of the basic equation of liquid velocity.* In figure 3, if a constant static pressure is assumed in a given cross-section, the following equations are obtained from the force balances for two control surfaces, i.e. ABCD and AEFD:

for surface ABCD,

$$y_m\Delta p - \rho g\int_0^{y_m}(1 - \alpha)\,\mathrm{d}y\,\Delta x = \tau_w\Delta x$$

for surface AEFD,

$$(y_m - y)\Delta p - \rho g\int_y^{y_m}(1 - \alpha)\,\mathrm{d}y\,\Delta x = \tau\Delta x$$

where $\Delta p$ is the difference of static pressures between two cross-sections AB and CD at an interval $\Delta x$. In the foregoing, momentum changes between the two cross-sections are

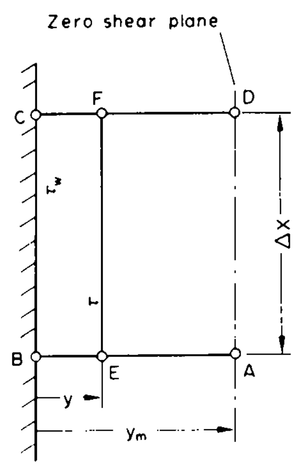


Figure 3. Parameters for force balances.

ignored and the weight of gas is neglected in comparison with the liquid. Eliminating the static pressure difference $\Delta p$ from the above equations, the shear stress is expressed as

$$\tau = \left(1 - \frac{y}{y_m}\right)\tau_w - \left(1 - \frac{y}{y_m}\right)\rho g \int_0^{y_m} \alpha\,\mathrm{d}y + \rho g \int_y^{y_m} \alpha\,\mathrm{d}y. \qquad [13]$$

The shear stress evaluated by this equation should equal the turbulent shear stress given by [11]. Hence

$$\left(1 - \frac{y}{y_m}\right)\tau_w - \left(1 - \frac{y}{y_m}\right)\rho g \int_0^{y_m} \alpha\,\mathrm{d}y + \rho g \int_y^{y_m} \alpha\,\mathrm{d}y = \rho\varepsilon_{TP}\frac{\mathrm{d}u}{\mathrm{d}y}. \qquad [14]$$

Here the following dimensionless variables are employed

$$u^* = \sqrt{(\tau_w/\rho)}, \quad r^* = 1 - y/y_m, \quad \phi = u/u^*. \qquad [15]$$

Applying [15] to [14], and rearranging, the basic equation of liquid velocity is written in dimensionless form:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r^*} = -\left\{\left(1 - B_1\int_0^1 \alpha\,\mathrm{d}r^*\right)r^* + B_1\int_0^{r^*} \alpha\,\mathrm{d}r^*\right\}\Big/\left(\frac{\varepsilon_{TP}}{y_m u^*}\right) \qquad [16]$$

where

$$B_1 = g y_m/u^{*2}. \qquad [17]$$

In the case of single-phase liquid flow there is no void at any location in a channel, i.e. $\alpha = 0$. Then [16] is reduced to Reichardt's equation for the velocity of turbulent flow.

Equation [13] is expressed using the dimensionless variables defined by [15] and [17] as:

$$\frac{\tau}{\rho u^{*2}} = \left(1 - B_1\int_0^1 \alpha\,\mathrm{d}r^*\right)r^* + B_1\int_0^{r^*} \alpha\,\mathrm{d}r^* \qquad [18]$$

which gives a dimensionless shear stress distribution. This corresponds to the numerator of [16]. Equation [12], on the other hand, is written as:

$$\varepsilon_{TP}^* = \frac{\varepsilon_{TP}}{y_m u^*} = (1-\alpha)\left[\frac{\kappa}{6}(1 - r^{*2})(1 + 2r^{*2}) + \kappa_1\alpha\frac{\hat{d}_B}{2y_m}\frac{\hat{U}_B}{u^*}\right] \qquad [19]$$

which gives a dimensionless apparent eddy diffusivity distribution and the denominator of [16].

In the case of a horizontal channel, gravity terms can be neglected in [18], thus the shear stress distribution becomes

$$\tau/\rho u^{*2} = r^*. \qquad [20]$$

It is, therefore, possible to derive the basic equation of liquid velocity from [16] by substituting $r^*$ for the right-hand numerator.

### 2.2 *Flow in circular pipes*

The preceding considerations concerning a two-dimensional flow in a channel with parallel walls are applicable to the case of a circular pipe. Here [11] and [12] can be written as:

$$\tau = -\rho\varepsilon_{TP}\frac{\mathrm{d}u}{\mathrm{d}r} \tag{21}$$

$$\varepsilon_{TP} = (1-\alpha)\left[\frac{\kappa R}{6}\sqrt{\left(\frac{\tau_w}{\rho}\right)}\left\{1-\left(\frac{r}{R}\right)^2\right\}\left\{1+2\left(\frac{r}{R}\right)^2\right\}+\kappa_1\alpha\left(\frac{\hat{d}_B}{2}\right)\hat{U}_B\right] \tag{22}$$

where $r$ is the radius at a location measured from the center line and $R$ the pipe radius.

Denoting the shear stress acting on the cylindrical control surface with a given radius $r$ by $\tau$, and the one acting on the pipe wall by $\tau_w$, the following force balances are obtained under the same assumptions as stated for parallel walls

$$\pi r^2\Delta p - \rho g\int_0^r (1-\alpha)2\pi r\,\mathrm{d}r\,\Delta x = 2\pi r\,\Delta x\tau$$

$$\pi R^2\Delta p - \rho g\int_0^R (1-\alpha)2\pi r\,\mathrm{d}r\,\Delta x = 2\pi R\,\Delta x\tau_w.$$

These two equations give the shear stress distribution as follows

$$\tau = \frac{1}{r}\left[\left(\frac{r}{R}\right)^2\left\{R\tau_w - \rho g\int_0^R \alpha r\,\mathrm{d}r\right\} + \rho g\int_0^r \alpha r\,\mathrm{d}r\right].$$

Correspondingly, the basic equation of liquid velocity can be described in dimensionless form as:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r^*} = -\left\{\left(1 - B_2\int_0^1 \alpha r^*\,\mathrm{d}r^*\right)r^* + B_2\frac{1}{r^*}\int_0^{r^*}\alpha r^*\,\mathrm{d}r^*\right\}\Big/\varepsilon_{TP}^* \tag{23}$$

where

$$\varepsilon_{TP}^* = \varepsilon_{TP}/Ru^*, \quad r^* = r/R, \quad B_2 = gR/u^{*2}.$$

### 2.3 *Equations for numerical calculation*

In bubbly flow, the measured void fraction profile across a flow channel is not simple enough to be fitted by a parabolic or power-law curve. Therefore, in order to calculate the corresponding liquid velocity distribution using the basic equation ([16] or [23]) it is practical to carry it out numerically. Then a profile of void fraction must be reduced to the stepwise one as shown in figure 4. Numbering the variables relating to each layer as $\phi_1, \phi_2, \ldots$ and $\alpha_1, \alpha_2, \ldots$, and integrating the basic equation in succession from a zero shear plane to the dimensionless distance $r^*$, the dimensionless liquid velocity $\phi_i$ at the $i$th layer is expressed as follows.

For upward flow in parallel walls:

$$\phi_i = P_iX_i(r^*) + Q_iY_i(r^*) + \phi_{i-1}(r_i^*), \quad \phi_0 = \phi_m \tag{24}$$

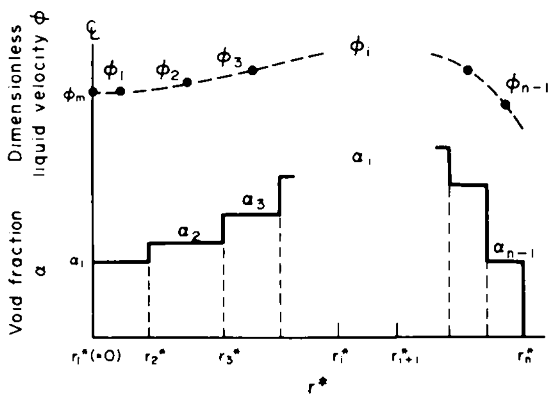


Figure 4. Correspondence of dimensionless liquid velocities to void fractions approximated in a steplike arrangement.

and for upward flow in a circular pipe:

$$\phi_i = R_i X_i(r^*) + S_i Z_i(r^*) + \phi_{i-1}(r_i^*)^*, \quad \phi_0 = \phi_m \tag{25}$$

where

$$X_i(r^*) = \int \frac{r^* \, dr^*}{2r^{*4} - r^{*2} - C_i} = \frac{1}{2\sqrt{D_i}} \ln \left| \frac{4r^{*2} - 1 - \sqrt{D_i}}{4r^{*2} - 1 + \sqrt{D_i}} \right| \tag{26}$$

$$\begin{aligned} Y_i(r^*) &= \int \frac{dr^*}{2r^{*4} - r^{*2} - C_i} \\ &= \frac{4}{\sqrt{D_i}} \left\{ \frac{1}{4(\sqrt{D_i} + 1)^{1/2}} \ln \left| \frac{2r^* - (\sqrt{D_i} + 1)^{1/2}}{2r^* + (\sqrt{D_i} + 1)^{1/2}} \right| \right. \\ &\quad \left. - \frac{1}{2(\sqrt{D_i} - 1)^{1/2}} \tan^{-1} \frac{2r^*}{(\sqrt{D_i} - 1)^{1/2}} \right\} \end{aligned} \tag{27}$$

$$\begin{aligned} Z_i(r^*) &= \int \frac{dr^*}{r^*(2r^{*4} - r^{*2} - C_i)} \\ &= \frac{1}{C_i} \left[ \frac{1}{4\sqrt{D_i}} \left\{ \sqrt{D_i} \ln \left| r^{*4} - \frac{1}{2} r^{*2} - \frac{C_i}{2} \right| \right. \right. \\ &\quad \left. \left. + \ln \left| \frac{4r^{*2} - 1 + \sqrt{D_i}}{4r^{*2} - 1 - \sqrt{D_i}} \right| \right\} - \ln r^* \right] \end{aligned} \tag{28}$$

$$P_i = \frac{1 - B_1 \int_0^1 \alpha \, dr^* + B_1 \alpha_i}{\frac{\kappa}{6}(1 - \alpha_i)} = \frac{1 - B_1(A_1 - \alpha_i)}{\frac{\kappa}{6}(1 - \alpha_i)} \tag{29}$$

$$Q_i = B_1 \frac{A_1 - \left\{\alpha_i r_i^* + \sum_{j=1}^{i-1} \alpha_j (r_{j+1}^* - r_j^*)\right\}}{\frac{\kappa}{6}(1 - \alpha_i)} \quad [30]$$

$$R_i = \frac{1 - B_2 \int_0^1 \alpha r^* \, \mathrm{d}r^* + B_2 \alpha_i}{\frac{\kappa}{6}(1 - \alpha_i)} = \frac{1 - B_2(A_2 - \alpha_i)}{\frac{\kappa}{6}(1 - \alpha_i)} \quad [31]$$

$$S_i = \frac{B_2}{2} \frac{\sum_{k=1}^{i-2} \alpha_{k+1}(r_{k+1}^{*2} - r_k^{*2}) - \alpha_i r_{i-1}^{*2}}{\frac{\kappa}{6}(1 - \alpha_i)} \quad (i \geqq 2). \quad [32]$$

In these equations

$$\left.\begin{aligned} A_1 &= \int_0^1 \alpha \, \mathrm{d}r^* \simeq \sum_{i=1}^{n-1} \alpha_i (r_{i+1}^* - r_i^*) \\ A_2 &= \int_0^1 \alpha r^* \, \mathrm{d}r^* \simeq \sum_{i=1}^{n-1} \frac{\alpha_i}{2}(r_{i+1}^{*2} - r_i^{*2}) \\ B_1 &= g y_m / u^{*2}, \quad B_2 = gR/u^{*2} \\ C_i &= 1 + 6\frac{\kappa_1}{\kappa}\left(\frac{\hat{d}_B}{2y_m}\right)\left(\frac{\hat{U}_B}{u^*}\right)\alpha_i \\ D_i &= 1 + 8C_i. \end{aligned}\right\} \quad [33]$$

Here consideration will be given to a case where the values of liquid flow rate, void fraction profile, $y_m$ or $R$, $\hat{d}_B$, $\hat{U}_B$ and $u^*$ are known in advance. The calculation procedures to obtain the liquid velocity profile are as follows:

(1) $\kappa_1$ is taken as unity (as for its value, a discussion is presented in the next section). Reduce the given void fraction profile to a stepwise one.

(2) Corresponding coefficients of $P_i$, $Q_i$, $R_i$ and $S_i$ to the $i$th layer are obtained using [29]–[33].

(3) A value of $\phi_m$ is assumed, and then the liquid velocity is calculated in turn from zero shear plane to the wall. The procedure of this stage is repeated until the calculated liquid flow rate agrees with the prescribed value.

## 3. DISCUSSION

The present analysis is applied to real bubbly flow. For this purpose, experiments were performed in air–water mixtures which flowed upward in a 34.8 mm inner diameter pipe (smooth surface, 3 m in total length). The experiments yielded information on velocity, void fraction distributions, and wall shear stress. The experimental procedure was as follows.

Air was injected continuously into a fully developed turbulent water stream through 60 holes of 0.3 mm diameter which were perforated on the periphery of the pipe wall. The two-phase bubbly mixtures then flowed upward in the test section. Measurements were carried out in a downstream section 0.6 m from the mixing point. The liquid velocity was determined by an impact pressure probe method which will be described briefly later. The void fraction was measured with the aid of an electrical resistivity probe. The wall shear stress $\tau_w$ was determined from the measured frictional pressure drop. The diameter of the bubbles were also estimated from photographs taken under a similar flow condition in another 25 mm × 50 mm transparent rectangular duct. The relative velocity of bubbles to the surrounding water was assumed to be equal to the terminal velocity in still water (Sato *et al.* 1973).

Typical examples of the results obtained both analytically and experimentally are presented in figures 5a and 6a, where the ordinate is the ratio of the local to the center line velocity $u/u_m$, while the abscissa is the dimensionless radial distance, $r^* = r/R$. The solid

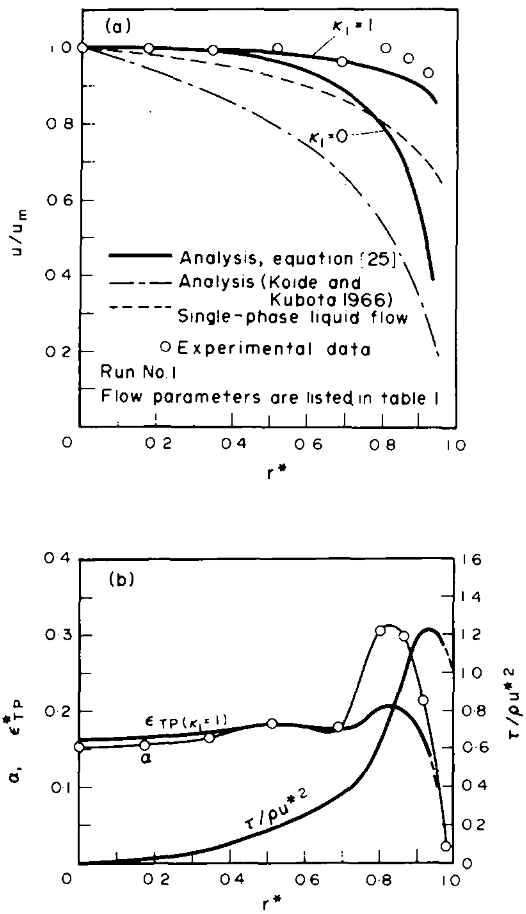


Figure 5. Profiles of liquid velocity, void fraction, shear stress and apparent eddy diffusivity in a 34.8 mm i.d. pipe (run 1).

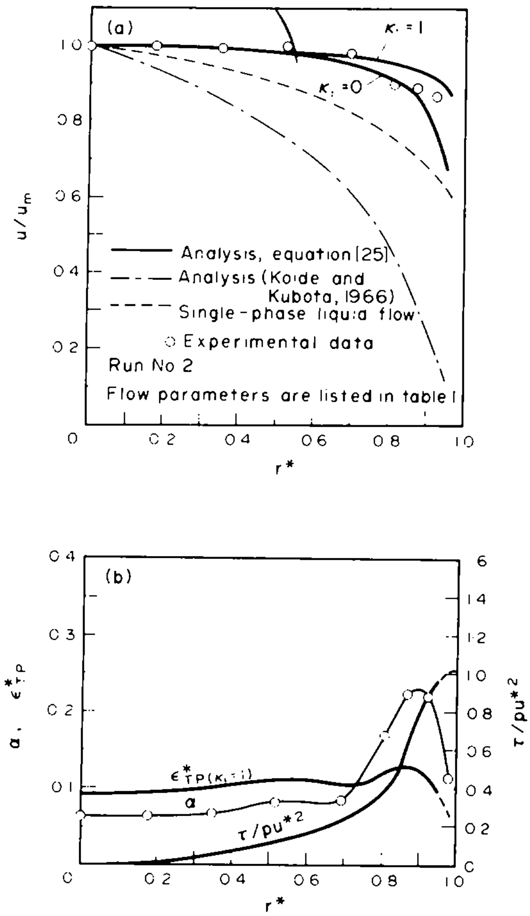


Figure 6. Profiles of liquid velocity, void fraction, shear stress and apparent eddy diffusivity in a 34.8 mm i.d. pipe (run 2).

lines in each figure represent the analytical prediction based upon this theory, where $\kappa_1 = 0$ and $\kappa_1 = 1$. The former indicates no effect of bubble agitation. The appropriate value of $\kappa_1$ is not yet known. Therefore, in the latter case, it is chosen so as to fit closely with the data points plotted by the open circles. The figures also include the liquid velocity profiles predicted by Koide & Kubota (1966) (dot-dashed line) and those of single-phase turbulent flow which corresponds to the same mass flow rate (dashed line).

The measured void fraction distributions are represented in figures 5b and 6b. In addition, the corresponding profiles of the dimensionless shear stress, $\tau/\rho u^{*2}$, and apparent eddy diffusivity, $\varepsilon^*_{TP} = (1 - \alpha)(\varepsilon' + \varepsilon'')/Ru^*$, are displayed in the figures. Principal flow parameters on the flows are tabulated in table 1.

Each void fraction distribution presented here shows a relatively flat profile in the core region ($r^* < 0.7$), and a protuberant curve with a maximum value near the wall. This might be attributed to the existence of a bubble sublayer near the wall. Consistent with

Table 1. Flow parameters for the experiments shown in figures 5 and 6

| Run No. | $w_{lo}^*$ (m/s) | $w_{go}^*$ (m/s) | Temp. (°C) | System press. (bar) | $\hat{d}_B$ (mm) | $\hat{U}_B$ (m/s) | $\hat{\alpha}$ | $\tau_w$ (Pa) |
|---|---|---|---|---|---|---|---|---|
| Run 1 | 0.50 | 0.175 | 13.0 | 1.38 | 4.2 | 0.23 | 0.192 | 3.44 |
| Run 2 | 1.00 | 0.207 | 22.0 | 1.38 | 3.4 | 0.25 | 0.111 | 5.69 |

* $w_{lo}$ and $w_{go}$ are the superficial velocities of water and air, referring to the flow of each phase alone.

this distribution, the shear stress first increases gradually with radial distance $r^*$ in the core. It then begins to increase steeply at $r^* = 0.7 \sim 0.8$. It is also seen from the figures that the apparent eddy diffusivity $\varepsilon_{TP}^*$, remains almost constant in the core of $r^* < 0.8$, but, in the vicinity of the wall ($r^* > 0.8$), it decreases with the increase of radial distance.

In accordance with the distribution of both shear stress and eddy diffusivity, liquid velocity can be determined by the present theory. The results suggest that its profile is rather flat compared with the profile for single-phase flow in the core, whereas it presents a steep decrease with increasing radius in the wall region. The comparison between both the predicted results for $\kappa_1 = 0$ and 1 implied that the greater is the value of $\kappa_1$, the flatter is the velocity profile.

On the other hand, because of the assumption that shear stress is uniform and therefore equal to the value at the wall ($\tau = \tau_w$), Koide & Kubota's (1966) analysis gives the largest velocity gradient in the core.

As mentioned previously, the liquid velocity was measured with the aid of an impact pressure probe, and was determined by the following equation:

$$u = \sqrt{\left(\frac{2}{(1-\alpha^2)}\frac{\Delta p}{\rho}\right)} \quad \text{(continuous squeeze model)} \qquad [34]$$

in which $\Delta p$ is the impact pressure and $\alpha$ is the void fraction at the point where the tip of the impact pressure probe is located. Agreement between the experimental values calculated using [34], and those predicted by the theory where $\kappa_1 = 1$ is remarkably good in the core, but becomes somewhat poor near the wall. Although the applicability of [34] was examined by Shires & Riley (1966), it seems that its validity in any bubble flow has not been completely established to date. Thus, uncertainty may be allowed, to some extent, in the reduction of measured values ($\Delta p$, $\alpha$, etc.) to the liquid velocity, especially in the vicinity of the wall.

## 4. CONCLUSIONS

An attempt was made to introduce a theoretical relationship between the profiles of liquid velocity and void fraction. It is necessary to establish a momentum equation for two-phase bubble flow, in particular a shear stress which might be connected closely with velocity fluctuations in a liquid phase.

In the present theory, the velocity fluctuations consist of two such components, one due to the inherent liquid turbulence independent of the existence of bubbles, ($u'$, $v'$), and the other due to the additional liquid turbulence caused by bubble motion ($u''$, $v''$).

An analytical relationship for evaluating the additional turbulence was derived, and a basic equation for determining the liquid velocity profile was developed.

Typical experimental results were employed for examining the validity of this theory. The agreement between experimental and theoretical results is remarkably good in the core, but becomes somewhat poor near the wall. To get further confirmation, many more detailed measurements on liquid velocity will be needed. In addition, an analysis should be developed to make clear the flow mechanism very close to the wall, i.e. within the bubble sublayer.

*Acknowledgements*—The authors wish to thank Mr. T. Honda and Mr. H. Okabe, who assisted with the measurements and helped with the preparation of this manuscript.

## REFERENCES

BANKOFF, S. G. 1960 A variable density single-fluid model for two-phase flow with particular reference to steam–water flow. *Trans. ASME, Ser. C* **82**, 265–272.

BEATTIE, D. R. H. 1972 Two-phase flow structure and mixing length theory. *Nucl. Enging Design* **21**, 46–64.

BROWN, F. C. & KRANICH, W. L. 1968 A model for the prediction of velocity and void fraction profiles in two-phase flow. *A.I.Ch.E.J.* **14**, 750–758.

KOIDE, K. & KOBOTA, H. 1966 Gas holdup distribution and liquid velocity distribution on bubble flow in vertical column (Japanese), *J. Soc. Chem. Engrs, Japan* **30**, 806–813.

LEVY, S. 1963 Prediction of two-phase pressure drop and density distribution from mixing length theory. *Trans. ASME, Ser. C* **85**, 137–152.

MILNE-THOMSON, L. M. 1968 *Theoretical Hydrodynamics*, p. 243. Macmillan.

REICHARDT, H. 1951 Vollständige Darstellung der turbulenten Geschwindigkeitsverteilung in glatten Leitungen. *ZAMM* **31**, 208–219.

SATO, Y., HONDA, T., SARUNATARI, S. & SEKOGUCHI, K. 1973 Air bubble velocity in Water streams in a vertical duct (Japanese), Preprint, 10th Symp. Heat Transfer Society of Japan, 5–8.

SEKOGUCHI, K., KAWAKAMI, Y. & NISHIKAWA, K. 1972a Simulation of forced-flow boiling with air–water two-phase fluid (Japanese), Preprint, 9th Symp. Heat Transfer Society of Japan, 21–24.

SEKOGUCHI, K., SATO, Y. & HONDA, T. 1972b An experimental investigation of air bubble motion in water streams in a vertical duct (Japanese), Preprint, 9th Symp. Heat Transfer Society of Japan, 431–434.

SHIRES, G. L. & RILEY, P. J. 1966 The measurement of radial voidage distribution in two-phase flow by isokinetic sampling, *AEEW-M650*, 1–26.

## APPENDIX

Two points $P_1(-\infty, y_1)$ and $P_2(0, y_2)$ are located on a given stream line in a non-viscous fluid stream past a cylinder of radius $a$, as shown in figure 7. Fluid which flows through point $P_1$ is shifted over a distance:

$$y_2 - y_1 = a\left\{\left[\left(\frac{y_1}{2a}\right)^2 + 1\right]^{\frac{1}{2}} - \left(\frac{y_1}{2a}\right)\right\} \tag{35}$$

towards the $y$-direction near the cylinder.

Let us apply the above equation to the drift of liquid due to bubble motion in a two-phase bubbly flow under the assumption of a two-dimensional non-viscous flow. As illustrated in figure 2 already, when a cylindrical bubble of diameter $d$ passes at a distance $\eta$ from the control plane, a liquid particle on it may be shifted in $y$-direction by $d_B\{\sqrt{((\eta/d_B)^2 + 1)} - (\eta/d_B)\}/2$. Then, if we assume that the distance over which liquid retaining its original momentum is displaced by a bubble in the transverse direction is proportional to the above value, the mixing length due to the bubble can be described as:

$$l_B = \text{const}\, d_B(\sqrt{(\eta^{*2} + 1)} - |\eta^*|) \tag{36}$$

where $\eta^*$ is a dimensionless distance defined as $\eta^* = \eta/d_B$.

By analogy with the single-phase flow, the magnitude of the $x$-component of velocity fluctuation on the control plane can be written as:

$$\begin{aligned} |u''|_{\eta^*} &= \text{const}\, l_B\left|\frac{du}{dy}\right| \\ &= \text{const}\, d_B(\sqrt{(\eta^{*2} + 1)} - |\eta^*|)\left|\frac{du}{dy}\right| \end{aligned} \tag{37}$$

while that of the $y$-component can be assumed as (Milne-Thomson 1968)

$$|v''|_{\eta^*} = \left\{\begin{array}{ll} \text{const}\, U_B & |\eta^*| < 1 \\ \text{const}\, U_B \dfrac{1}{\eta^{*2}} & |\eta^*| \geqq 1. \end{array}\right\} \tag{38}$$

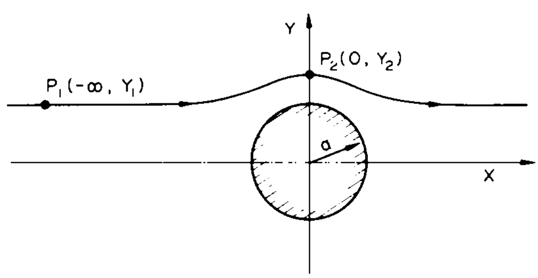


Figure 7. Non-viscous fluid flow past a cylinder.

Now, let us introduce the following stochastic discrete function

$$\sigma = \begin{cases} 1 & \text{(liquid phase exists at a point)} \\ 0 & \text{(gas phase does).} \end{cases} \qquad [39]$$

When bubbles pass temporarily at random in sequence at a distance $\eta$ from the control plane, the time-averaged value of the product of such velocity fluctuations should be considered on the plane. This can be expressed by using the above function in connection with the local void fraction as follows:

$$\begin{aligned} |u''v''|_{\eta^*} &= \frac{1}{T}\int_0^T |u''|_{\eta^*}|v''|_{\eta^*}\{1 - \sigma(\eta^*)\}\,\mathrm{d}t \\ &= |u''|_{\eta^*}|v''|_{\eta^*}\alpha(\eta^*) \end{aligned} \qquad [40]$$

where

$$\alpha(\eta^*) = \overline{1 - \sigma(\eta^*)}. \qquad [41]$$

In fact, a large number of bubbles pass at various distances from the control plane. The velocity fluctuations generated by these bubbles should be superposed. Thus, assuming that [40] can be used independently of the void properties across the flow, the product of velocity fluctuations averaged with respect to time and distance is expressed as:

$$-\overline{u''v''} = \left[\int_{\text{cross-section}} |u''|_{\eta^*}|v''|_{\eta^*}\alpha(\eta^*)\,\mathrm{d}\eta^*\right] \qquad [42]$$

in which the signs of $u''$ and $v''$ are considered.

Substituting [37] and [38] into [42], the following equation is obtained

$$\begin{aligned} -\overline{u''v''} = \Bigg[&\text{const}\int_{-1}^{1} d_B U_B(\sqrt{(\eta^{*2}+1)} - |\eta^*|)\alpha\,\mathrm{d}\eta^* \\ &+ \text{const}\left\{\int_{y_1/d_B}^{-1} d_B U_B \frac{\sqrt{(\eta^{*2}+1)} - |\eta^*|}{\eta^{*2}}\alpha\,\mathrm{d}\eta^* \right. \\ &\left. + \int_1^{(2R-y_1)/d_B} d_B U_B \frac{\sqrt{(\eta^{*2}+1)} - |\eta^*|}{\eta^{*2}}\alpha\,\mathrm{d}\eta^*\right\}\Bigg]\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right|. \end{aligned} \qquad [43]$$

Furthermore, if a linear void fraction distribution is assumed near the control plane, and $d_B$ and $U_B$ are replaced by their spatial mean values in the cross-section, $\hat{d}_B$ and $\hat{U}_B$, the correlation can be approximately expressed by:

$$-\overline{u''v''} = \text{const}\,\alpha\hat{d}_B\hat{U}_B\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right| \qquad [44]$$

where $\alpha$ is the void fraction on the control plane.

The hypothesis of [6] in the present analysis is based upon the above argument.

**Résumé** Un traitement analytique amélioré est développé qui rend facile la prévision satisfaisante de la distribution de vélocité liquide dans un écoulement de bulle en deux phases.

Dans l'analyse, la contrainte de cisaillement dans la phase liquide est considérée être importante. Lorsque la variation de la vélocité turbulente est subdivisible en deux composants: l'un dû à la turbulence inhérente du liquide indépendemment de l'existence d'une bull, $(u', v')$ et l'autre dû à la turbulence supplémentaire du liquide par l'agitation de la bulle, $(u'', v'')$, il est possible de séparer la contrainte de cisaillement en deux parties, $-\rho u'v'$ et $-\rho u''v''$ correspondant à $(u', v')$ et à $(u'', v'')$ respectivement.

Une équation fondamentale pour la distribution de vélocité liquide est dérivée à partir d'un développement approfondi de ce traitement. L'accord entre les profils de vélocité mesuré et calculé est assez bon, surtout dans la région du noyau d'une conduite.

**Auszug** Es wird eine verbesserte analytische Behandlung entwickelt, welche die befriedigende Voraussage der Flüssigkeit-Geschwindigkeitsverteilung in zweiphasiger Blasenströmung möglich macht.

In der Analyse wird die Scherspannung in der Flüssigkeitsphase als wichtig angesehen. Wenn die Schwankung von Turbulenzgeschwindigkeit in zwei Bestandteile unterteilt werden kann, einer, infolge der eigenen Flüssigkeitsturbulenz, unabhängig von dem Vorhandensein der Blase, $(u' v')$, und der andere infolge der zusätzlichen Flüssigkeitsturbulenz, $(u'', v'')$, ist es möglich, die Scherspannung in zwei Komponenten aufzuteilen, $-\rho u'v'$ und $-\rho u''v''$ entsprechend $(u'v')$, beziehungsweise $(u''v'')$.

Eine Grundgleichung der Flüssigkeit-Geschwindigkeitsverteilung wird aus weiterer Entwicklung dieser Behandlung abgeleitet. Die Übereinstimmung zwischen den gemessenen Geschwindigkeitsprofilen und den berechneten ist ziemlich eng, besonders in der Kerngegend eines Kanals.

**Резюме**—Проведено тщательное аналитическое исследование, дающее возможность удовлетворительно предсказывать распределение жидкости по скоростям в двухфазном потоке с пузырьковыми включениями.

При этом анализе, касательное напряжение в жидкой фазе считается важным. Если флуктуацию турбулентной скорости подразделить на две составляющие: одна – скорость присущая турбулентной жидкости независящая от пузырьковых включений, $(u', v')$, а другая – турбулентность вследствие возмущения пузырьками добавочной жидкости, $(u'', v'')$, касательное напряжение можно разделить на две составляющие, $-\rho u' v'$ и $-\rho u'' v''$, соответствующие $(u', v')$ и $(u'', v'')$.

Добавочным рассмотрением этого исследования получили основное уравнение на распределение жидкости по скоростям. При этом достигается удовлетворительное согласие между измеренными профилями скоростей и рассчитанными, особенно в центральной части трубы.